OPTIMAL CONTROL IN ONE-DIMENSIONAL ELASTIC-PLASTIC MODELS

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This work is devoted to investigating the problem of optimal control by initial data in one-dimensional elastic-plastic models. We consider dynamic elastic-plastic problems for rods and cylindrical shells with axial symmetry. We must select initial data such that the values of the solution at a given moment in time differ as little as possible from prescribed values. The existence of the solution for the problem formulated is proved. We construct a family of auxiliary problems and establish results on the convergence of the solutions. In dynamics problems it is possible to advance the initial data as the control function. An entire series of works is devoted to the problem of optimal control in the theory of elasticity (see, for example, [1, 2] and the bibliographies therein). The absence of similar results for elastic-plastic problems is explained by the difficulty in substantiating the validity of the initial models [3]. In this work, we present a possible formulation of dynamic elastic-plastic optimal control problems. We find the conditions on the external data and initial conditions for which the problem of optimal control has a solution.

1. Formulation of the problem of elastic-plastic deformation of a rod consists of the following [4-6]. In the region $Q = (a, b) \times (0, T)$ we must find functions u, w, n, m, ξ_1, ξ_2 satisfying

$$u_t - n_x = f_1,\ w_t - m_{xx} = f_2; \tag{1.1}$$

$$u_x = n_t + \xi_1,\ -w_{xx} = m_t + \xi_2; \tag{1.2}$$

$$\Phi(n,\ m) \leqslant 0; \tag{1.3}$$

$$\xi_1(\bar{n} - n) + \xi_2(\bar{m} - m) \leqslant 0\ \forall(\bar{n},\ \bar{m}),\ \Phi(\bar{n},\ \bar{m}) \leqslant 0; \tag{1.4}$$

$$u = u_0,\ w = w_0,\ n = n_0,\ m = m_0 \text{ for } t = 0; \tag{1.5}$$

$$n = m = m_x = 0 \text{ for } x = a,\ b. \tag{1.6}$$

Here Φ: $R^2 \to R$ is a given convex, continuous function which characterizes the transition to the plastic state; f_1, f_2 are external forces; u, w are the tangential and normal velocities of points on the rod; n, m are the internal force and the bending moment, respectively. The subscripts t and x denote differentiation. Equation (1.1) is the equation of motion; (1.2) represents the rates of deformation and curvature as the sum of elastic and plastic components; (1.3) signifies that the unknowns n and m do not go beyond the limits of the yield surface $\Phi(n, m) = 0$. The inequality (1.4) gives the direction of the vector (ξ_1, ξ_2) with respect to the yield surface and satisfies the principle of maximum dissipation rate.

First we will introduce some notation, and then we formulate the problem of optimal control by the initial data. Let $H(a, b) = H^1(a, b) \times H^2(a, b) \times H_0^1(a, b) \times H_0^2(a, b)$ ($H^s(a, b)$ and $H_0^s(a, b)$ are Sobolev spaces), $K = \{(n, m) | n, m \in L^2(a, b), \Phi(n(x), m(x)) \le 0\}$ is the set of admissible moments and internal forces, $W = \{(n, m) | n \in H_0^1(a, b), m \in H_0^2(a, b)\}$.

It can be shown (see [7]) that if $(0, 0) \in K$, f_i, $f_{it} \in L^2(Q)$, $V_0 \equiv (u_0, w_0, n_0, m_0) \in H(a, b)$, $(n_0, m_0) \in K$, then there exist unique functions u, w, n, m satisfying (1.1), initial conditions (1.5) and the inequality

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$$[n_t, \bar{n}-n]+[m_t, \bar{m}-m]+[w, \bar{m}_{xx}-m_{xx}]+[u, \bar{n}_x-n_x]\geqslant 0$$
$$\forall(\bar{n}, \bar{m})\in L^2(0, T; K\cap W). \tag{1.7}$$

$(n(t), m(t))$ does not leave the yield surface: $(n(t), m(t)) \in K$ almost everywhere on $(0, T)$. Here $[\cdot, \cdot]$ is the scalar product in $L^2(Q)$. Inequality (1.7) was obtained from (1.2) and (1.4) by eliminating ξ_1 and ξ_2, integrating by parts and using boundary conditions (1.6).

We formulate the problem of optimal control by initial data in the following way. Let a convex, closed, and bounded set $U \subset H(a, b)$ be given such that if $V_0 \equiv (u_0, w_0, n_0, m_0) \in U$, then $(n_0, m_0) \in K$. We denote the norm on the spaces $L^2(a, b)$ and $[L^2(a, b)]^4$ by $\|\cdot\|_0$, and the solution $V \equiv (u, w, n, m)$ at time T by $V(T)$. Let a function $V_* \in [L^2(a, b)]^4$ be given. We require initial data $V_0 \in U$ such that the deviation of $V(T)$ from V_* is minimized. In other words, we must solve the problem

$$\inf_{V_0\in U} J(V_0) \tag{1.8}$$

$(J(V_0) = \|V(T) - V_*\|_0)$. The following is true:

THEOREM 1. Let $(0, 0) \in K$, f_i, $f_{it} \in L^2(Q)$. Then there exists a solution to the optimal control problem (1.8).

We sketch the proof of this assertion. First of all it is necessary to establish an estimate of the solution based on the initial data. To do this, an auxiliary problem with penalties is examined and an estimate for it is established which is uniform in the penalty parameter. The concluding part of the proof is based on an analysis of a minimizing sequence. Let us now examine the auxiliary problem with penalties. Let ε be a positive parameter, $p = (p_1, p_2)$ be the penalty operator, associated with the set K and acting from $[L^2(a, b)]^2$ to $[L^2(a, b]^2$ [8]. In the region Q we must find functions u^ε, w^ε, n^ε, m^ε satisfying

$$u_t^\varepsilon - n_x^\varepsilon = f_1, \quad w_t^\varepsilon - m_{xx}^\varepsilon = f_2; \tag{1.9}$$
$$n_t^\varepsilon - u_x^\varepsilon + \frac{1}{\varepsilon} p_1(n^\varepsilon, m^\varepsilon) = 0, \quad m_t^\varepsilon + w_{xx}^\varepsilon + \frac{1}{\varepsilon} p_2(n^\varepsilon, m^\varepsilon) = 0 \tag{1.10}$$

with initial and boundary conditions

$$u^\varepsilon = u_0,\ w^\varepsilon = w_0,\ n^\varepsilon = n_0,\ m^\varepsilon = m_0 \text{ for } t = 0,$$
$$n^\varepsilon = m^\varepsilon = m_x^\varepsilon = 0 \text{ for } x = a, b.$$

An a priori estimate of the boundary problem formulated here is obtained in the following way. First we multiply (1.9) and (1.10) by u^ε, w^ε, n^ε, m^ε, respectively, and then differentiate the resultant equations by t and multiply, respectively, by u_t^ε, w_t^ε, n_t^ε, m_t^ε. In this case, terms containing the penalty operator are nonnegative [8]. Furthermore $V_t^\varepsilon(0)$ must be uniformly bounded in $[L^2(a, b)]^4$, as a consequence of (1.9) and (1.10). The differentiability of (1.9) and (1.10) by t can be established by enlisting the Galerkin method for proof of the existence of the solution. The final estimate has the form

$$\max_{0\leqslant t\leqslant T}\{\|V^\varepsilon(t)\|_0^2 + \|V_t^\varepsilon(t)\|_0^2\} \leqslant c(T, f_1, f_2, V_0).$$

Here the dependence of the constant c on T, f_1, f_2, and V_0 has been made explicit. Note that c does not depend on ε. In accordance with this estimate we select from the sequence V^ε, V_t^ε a subsequence which *-weakly converges in $L^2(0, T; [L^2(a, b)]^4)$ for $\varepsilon \to 0$. The limiting function V satisfies (1.1) and (1.7) with initial conditions (1.5), for which $(n(t), m(t)) \in K$. The following estimate holds:

$$\max_{0\leqslant t\leqslant T}\{\|V(t)\|_0^2 + \|V_t(t)\|_0^2\} \leqslant c(T, f_1, f_2, V_0). \tag{1.11}$$

Inequality (1.11) allows us to prove the solvability of the formulated optimal control problem (1.8). Let $V_0{}^i$ be a minimizing sequence. It is bounded in the space $H(a, b)$ and therefore by selecting the appropriate subsequence we can assume that $V_0{}^i \to V_0$ weakly in $H(a, b)$ and strongly in $[L^2(a, b)]^4$. Due to the weak closure of the set U we have $V_0 \in U$. Moreover, it can be shown that the solution to the problem V^i corresponding to $V_0{}^i$ also converges to V, with initial data V_0. The substantiation of this fact and the limiting transformation are based on an a priori estimate of the form (1.11). The concluding part of the proof uses the property of weak semicontinuity from below of the functional

$$d \equiv \inf_{V_1 \in U} J(V_1) = \underline{\lim}\, J(V_0^i) = \underline{\lim}\, \|V^i(T) - V_*\|_0 \geqslant \|V(T) - V_*\|_0 \geqslant d.$$

It follows that the initial value V_0 is such that the difference $V(T) - V_*$ is minimal.

2. We examine the case of an elastic-plastic shell. Let the set K be defined as in Sec. 1, using the function Φ and $W = \{(n, m) | n \in L^2(a, b), m \in H_0{}^2(a, b)\}$. The statement of the problem of elastic-plastic deformation of a cylindrical shell with axial symmetry consists of the following. In the region Q we must find functions w, n, m, ξ_1, ξ_2 satisfying

$$w_t - m_{xx} - n = f; \tag{2.1}$$

$$-w = n_t + \xi_1, \; -w_{xx} = m_t + \xi_2; \tag{2.2}$$

$$\Phi(n, m) \leqslant 0; \tag{2.3}$$

$$\xi_1(\bar{n} - n) + \xi_2(\bar{m} - m) \leqslant 0 \;\forall(\bar{n}, \bar{m}), \; \Phi(\bar{n}, \bar{m}) \leqslant 0, \tag{2.4}$$

and also satisfying th einitial and boundary conditions

$$w = w_0, \; n = n_0, \; m = m_0 \text{ for } t = 0; \tag{2.5}$$

$$m = m_x = 0 \text{ for } x = a, b \tag{2.6}$$

(w is the rate of normal deflection, m is the bending moment, n is the circumferential force).

The solvability of (2.1)-(2.6) can be proved [7]. In this case there exist functions w, n, m satisfying (2.1), initial conditions (2.5) and the inequality

$$[m_t, \bar{m} - m] + [n_t, \bar{n} - n] + [w, \bar{m}_{xx} - m_{xx}] + [w, \bar{n} - n] \geqslant 0$$

$$\forall(\bar{n}, \bar{m}) \in L^2(0, T; K \cap W), \; (n(t), m(t)) \in K.$$

The problem of optimal control by initial data consists of selecting from amongst the elements of the closed, convex, and bounded set $U \subset H^2(a, b) \times L^2(a, b) \times H_0^2(a, b)$ those elements such that the difference between the values of the solution at time T and the given function $V_* \in [L^2(a, b)]^3$ is minimized. In other words, we must find the solution to

$$\inf_{V_0 \in U} \|V(T) - V_*\|_0, \tag{2.7}$$

where V corresponds to the initial data V_0.

THEOREM 2. Let $(0, 0) \in K$, f, $f_t \in L^2(Q)$, and the set U be such that if $(w_0, n_0, m_0) \in U$, then $(n_0, m_0) \in K$. Then a solution to the optimal control problem (2.7) exists.

The outline for the proof is the same as that for Theorem 1. We first examine the auxiliary problem with penalties and establish an a priori estimate of the solution using the initial data, and then carry out a transformation in the limit of the penalty parameter. Then in conclusion we construct a sequence of initial values which converges to the solution of (2.7).

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THEORY OF PLASTIC DEFORMATION FOR MULTICOMPONENT POROUS MEDIA

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The creation of dispersed-reinforced materials having specific technical properties is achieved by the bonding of heterogeneous metals through plastic deformation of powdered mixtures. The properties of the composites formed in this way are qualitatively distinguished from those of the component materials. To a significant degree this is due to the presence of pores. Theoretical models of the plastic deformation of porous media can be used in the choice of methods and regimes of pressure moulding employed to obtain quality manufactured products. In this work we investigate the features of plastic deformation of porous media containing dispersed inclusions. A method is employed which gives an approximate expression for the composite dissipation function [1-8]. We obtain the conditions under which the inclusions behave as rigid particles or deform together with the matrix.

1. We examine a rigid-plastic material made up of a connecting matrix with a uniform distribution of inclusions and pores in it. The matrix and inclusions satisfy the von Mises condition with plastic flow limits k_0 and k_1, respectively. The problem consists of constructing approximate expressions for the composite dissipation function $D^*(\langle\varepsilon_{ij}\rangle)$, which in combination with an associated stress rule $\langle\sigma_{ij}\rangle = \partial D^*/\partial\langle\varepsilon_{ij}\rangle$ determines the plasticity conditions [1-8]. Here σ_{ij}, ε_{ij} are components of the stress and plastic strain rate tensors, and the angular brackets denote averaging of the field over the material volume.

The dissipation function of the macroscopic medium $D^*(\langle\varepsilon_{ij}\rangle)$ is obtained as the minimum value of the dissipation rate in a unit of macroscopic volume V of the porous body:

$$D = \frac{1}{V}\int\limits_{V_0} k_0 \sqrt{\varepsilon_{ij}\varepsilon_{ij}}\, dV + \frac{1}{V}\int\limits_{V_1} k_1 \sqrt{\varepsilon_{ij}\varepsilon_{ij}}\, dV \tag{1.1}$$

($V = V_S + V_2$, where the solid-phase volume is $V_S = V_0 + V_1$; V_0, V_1, and V_2 are the volumes of the matrix, inclusions, and pores, respectively).

By representing the integral over V_0 in the form of the difference between the integrals over V_S and V_1, we have the functional

$$D = k_0\langle\sqrt{\varepsilon_{ij}\varepsilon_{ij}}\rangle_S - (k_0 - k_1)\,\langle\sqrt{\varepsilon_{ij}\varepsilon_{ij}}\rangle_1, \tag{1.2}$$

which for $k_1 = k_0$ reduces to the expression for the dissipation function of a porous body with a homogeneous solid phase [2]. The indices after the angular brackets in (1.2) signify averaging over the appropriate phase.

Following [2-7], we employ the approximate relations

$$\langle\sqrt{\varepsilon_{ij}\varepsilon_{ij}}\rangle_S \approx \sqrt{\langle\varepsilon_{ij}\varepsilon_{ij}\rangle_S},\quad \langle\varepsilon_{ij}\varepsilon_{ij}\rangle_n \approx \langle\varepsilon_{ij}\rangle_n\langle\varepsilon_{ij}\rangle_n, \tag{1.3}$$

where n = 1, 2. Using $2\varepsilon_{ij} = v_{i,j} + v_{j,i}$, the value of $\langle\varepsilon_{ij}\rangle_2$ is determined by the displacement rate v_i at the pore surface according to the Gauss–Ostrogradskii formula. In

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